Inelastic Interaction and Non-Traveling-Wave Effects for Two Multi-Dimensional Burgers Models from Fluid Dynamics and Astrophysics with Symbolic Computation

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Describing the surface perturbations of a shallow viscous fluid, cosmic-ray-modified shock structures and electromagnetic waves in a saturated ferrite, the (2+1)- and (3+1)-dimensional Burgers equations are investigated in this paper. In view of the higher space dimensionality, the transformations from such two models to a (1+1)-dimensional Burgers equation are constructed with symbolic computation. Via the obtained transformations, three families of multi-dimensional N-shock-wave-like solutions are specially presented, which recover some previously published solutions. The inelastically interacting properties and some non-traveling-wave effects of shock waves are discussed through the figures for several sample solutions. Additionally, possible applications for those solutions and effects in some fields are also pointed out.

Key words: Multi-Dimensional Burgers Equations; Inelastic Interaction; Non-Traveling-Wave Effects; N-Shock-Wave-Like Solutions; Symbolic Computation.

1. Introduction

In dissipative media, the (1+1)-dimensional Burgers equation [(1+1)-DBE]

$$u_t + uu_x - \nu u_{xx} = 0, \tag{1}$$

where ν is a real constant, was first derived as a model for turbulent fluid flow in a channel [1] and the structure of a shock wave [2]. Subsequently, the (2+1)-dimensional Burgers equation [(2+1)-DBE]

$$(u_t + uu_x - u_{xx})_x + u_{yy} = 0 \tag{2}$$

and the (3+1)-dimensional Burgers equation [(3+1)-DBE]

$$(u_t + uu_x - u_{xx})_x + u_{yy} + u_{zz} = 0, \tag{3}$$

where y and z are two transverse coordinates, were deduced in many physical fields such as nonlinear acoustics [3], viscous fluids [4, 5], astrophysics [6], radiative magnetogasdynamics [7] and electromagnetics [8]. The (2+1)- and (3+1)-DBEs can describe the weakly nonlinear multi-dimensional shocks in the sense that the scale length of variation in the y and z directions is much larger than that in the x direction [6, 9]. Applications of these two multi-dimensional models could be seen from the following examples: in fluid dynamics, the (2+1)-DBE governs the surface perturbations of a shallow viscous fluid heated from below provided that the Rayleigh number of the system satisfies the condition $R \neq 30$ [5] (at the point $R = 30$, the dissipative coefficient equals zero and the system becomes predominantly dispersive); in astrophysics, the (2+1)- and (3+1)-DBEs can be used to describe weak multi-dimensional long-wavelength shocks modified by the first-order Fermi acceleration of energetic charged particles [6, 10]; in electromagnetics, the (2+1)-DBE is able to explain how dissipation and nonlinearity affect an electromagnetic perturbation propagating into a saturated ferromagnet in the presence of an external magnetic field [8, 11]. Specially, the (2+1)- or (3+1)-DBE is sometimes referred to as the Zabolotskaya-Khoklov equation [3, 9] in nonlinear acoustics, with the terms u_{yy} and u_{zz} representing wave diffraction.

Recently, much attention has been paid to the inelastic interaction of solitons [12–21] which is different from the elastic interaction in the sense of the interchange of amplitudes and redistribution of physical quantities for interacting solitons [12–17]. It has been found that a large class of integrable and nonintegrable nonlinear evolution equations (NLEEs) admit the inelastic collision properties, such as the Davey-Stewartson model [13], Kadomtsev-Petviashvili equation [14], N-coupled nonlinear Schrödinger equations [15], Zakharov system [16] and generalized Broer-Kaup system [17]. On the other hand, some current experiments reveal that the inelastic collisions have been observed for the topological solitons in a parallel array of Josephson junctions [18], three-dimensional bright spatial solitons in a saturable self-focusing medium [19] and incoherent screening spatial solitons in a strontium barium niobate crystal [20]. In addition, the experimental possibilities for observing the inelastic collisions of two-dimensional bright solitons in dipolar Bose-Einstein condensates have also been predicted [21].

As investigated in [8, 9, 11, 22], the interaction of solitary waves for the $(n+1)$-DBE ($n = 1, 2, 3$) is inelastic because their N-shock-wave solutions (which are constructed by the bilinear transformation method) describe the coalescence of N traveling shock waves into a single one. It is worth noting that the coalescence phenomena of shock waves have already been reported in many studies including the modulated waves in film flows [23], impurity concentration in localized structures in one-dimensional hydrodynamics [24], shock-like dynamics of inelastic gases [25], electromagnetic waves in a saturated ferrite [8, 11] and passive random walker dynamics on growing surfaces [26]. Therefore, it is not without practical importance to make further investigations on the (2+1)- and (3+1)-DBEs and relevant inelastic interaction of shock waves.

Although Painlevé analysis has shown that neither the (2+1)- nor the (3+1)-DBE is Painlevé-integrable [9], such two models have still attracted much interest in recent years because of their wide range of applications. By using the balancing-act method [27], further extended tanh method [28, 29] and generalized sine-Gordon equation expansion method [30], abundant analytic solutions of the (2+1)- and (3+1)-DBEs have been obtained, including the one-shock-wave-like solutions. Through the Lie's method of infinitesimal transformations, the symmetry classification has also been made for the (2+1)-DBE with a variable-coefficient function [31]. In the present paper, considering the higher space dimensionality, we intend to perform symbolic computation [27–30, 32] on the (2+1)- and (3+1)-DBEs and transform them to a (1+1)-DBE by an ansatz developed from the direct method in [33]. Since the properties and solutions of the (1+1)-DBE have been studied widely and deeply, as seen, e. g., in [22, 34], we can employ the obtained transformations to study the two multi-dimensional Burgers models on the basis of the (1+1)-DBE.

The outline of this paper is as follows: in Sections 2 and 3, with the aid of symbolic computation, we will construct the transformations from the (3+1)- and (2+1)-DBEs to a (1+1)-DBE, respectively; in Section 4, we will remark on the obtained transformations and present several families of multi-dimensional N-shock-wave-like solutions; in Section 5 will be our discussion and summary.

2. Transformations from (3+1)-DBE to (1+1)-DBE

In this section, we will demonstrate in detail how to construct the transformations from the (3+1)-DBE to a (1+1)-DBE as the computing process on the (3+1)-DBE is much more complicated than that on the (2+1)-DBE. To achieve our goal, the transformations are necessarily executed in two steps: (a) reduce the (3+1)-DBE to an once-differentiated Burgers equation (ODBE)

$$\left(w_\eta + w w_\zeta - w_{\zeta\zeta}\right)_\zeta = 0; \tag{4}$$

(b) integrate the ODBE and transform the resulting equation to a (1+1)-DBE.

2.1. Reduction from (3+1)-DBE to ODBE

In this subsection, with the aid of symbolic computation, we construct the transformations from (3) to (4) in the form

$$u = \alpha(x,y,z,t) + \beta(x,y,z,t) w[\zeta(x,y,z,t), \eta(x,y,z,t)], \tag{5}$$

where $\alpha(x,y,z,t)$, $\beta(x,y,z,t)$, $\zeta(x,y,z,t)$ and $\eta(x, y, z, t)$ are to be determined. Ansatz (5) has been developed from the direct method in [33] and successfully applied to seek the similarity reductions of some multi-dimensional NLEEs [35].

Requiring that $w(\zeta,\eta)$ should satisfy (4), ansatz (5) can be reduced to

$$u=\alpha(x,y,z,t) + \theta(y,z,t)w[\theta(y,z,t)x+\sigma(y,z,t),\eta(y,z,t)] \quad (6)$$

with

$$\alpha=-\frac{\theta(x\theta_t+\sigma_t)+(x\theta_y+\sigma_y)^2+(x\theta_z+\sigma_z)^2}{\theta^2}, \quad (7)$$

where $\theta(y,z,t)$, $\sigma(y,z,t)$ and $\eta(y,z,t)$ are determined by the following equations:

$$\eta_y^2+\eta_z^2=0, \quad (8)$$

$$2(\theta_y\eta_y+\theta_z\eta_z)x+\theta\eta_t+2\eta_y\sigma_y+2\eta_z\sigma_z=\theta^3, \quad (9)$$

$$(-2\theta_y^2-2\theta_z^2+\theta\theta_{zz}+\theta\theta_{yy})x -2\theta_y\sigma_y-2\theta_z\sigma_z+\theta\sigma_{yy}+\theta\sigma_{zz}=0, \quad (10)$$

$$2\eta_y\theta_y+2\eta_z\theta_z+\theta\eta_{yy}+\theta\eta_{zz}=0, \quad (11)$$

$$-2\theta_y^2-2\theta_z^2+\theta\theta_{yy}+\theta\theta_{zz}=0, \quad (12)$$

$$\alpha_{xt}+\alpha_x^2+\alpha\alpha_{xx}-\alpha_{xxx}+\alpha_{yy}+\alpha_{zz}=0. \quad (13)$$

It is easy to find that (8) actually corresponds to the linear advection equation

$$\eta_y+\mathrm{i}\varepsilon\eta_z=0 \quad \text{with} \quad \varepsilon=\pm1, \quad (14)$$

which has the general solution

$$\eta=P(\tau_\varepsilon,t), \quad \tau_\varepsilon=z-\mathrm{i}\varepsilon y, \quad (15)$$

where $P(\tau_\varepsilon,t)$ is an arbitrary complex differentiable function of τ_ε and t. Substitution of (15) into (11) leads to

$$\theta_y+\mathrm{i}\varepsilon\theta_z=0, \quad (16)$$

which gives the general solution for $\theta(y,z,t)$ of the form

$$\theta=Q(\tau_\varepsilon,t), \quad (17)$$

where $Q(\tau_\varepsilon,t)$ is an arbitrary complex differentiable function of τ_ε and t. By virtue of (17), we know that (12) is satisfied automatically, while (9) and (10) reduce to

$$P_tQ-2\mathrm{i}\varepsilon P_{\tau_\varepsilon}\sigma_y+2P_{\tau_\varepsilon}\sigma_z=Q^3, \quad (18)$$

$$(\sigma_{yy}+\sigma_{zz})Q-2(\sigma_z-\mathrm{i}\varepsilon\sigma_y)Q_{\tau_\varepsilon}=0. \quad (19)$$

We have to point out that it is very difficult to get the general solution for $\sigma(y,z,t)$ from the overdetermined set of equations (13), (18) and (19). According to (19), we consider two special cases as follows:

(A) $Q_{\tau_\varepsilon}=0$: In this case, $Q_{\tau_\varepsilon}=0$ indicates that Q is only dependent on t, i. e.,

$$Q=h(t), \quad (20)$$

where $h(t)$ is an arbitrary real differentiable function of t. For $h(t)\neq0$, (19) corresponds to the Laplace equation

$$\sigma_{yy}+\sigma_{zz}=0 \quad (21)$$

with the general solution

$$\begin{aligned}&\sigma=F(\tau_+,t)+G(\tau_-,t),\\ &\tau_+=z-\mathrm{i}y \;\text{ and }\; \tau_-=z+\mathrm{i}y,\end{aligned} \quad (22)$$

where $F(\tau_+,t)$ and $G(\tau_-,t)$ are two arbitrary complex differentiable functions. Symbolic computation on (13) and (18) yields

$$h(t)h''(t)-2h'^2(t)+16F_{\tau_+\tau_+}G_{\tau_-\tau_-}=0, \quad (23)$$

$$h(t)P_t+2(1-\varepsilon)P_{\tau_\varepsilon}F_{\tau_+}+2(1+\varepsilon)P_{\tau_\varepsilon}G_{\tau_-}=h^3(t). \quad (24)$$

Noting that $P(\tau_\varepsilon,t)$, $h(t)$, $F(\tau_+,t)$ and $G(\tau_-,t)$ are independent of one another before (23) and (24) appear, we could regard (23) and (24) as constraints among the four functions.

Now, collecting all the results we have obtained, the first transformation from (3) to (4) is presented in the form

$$\begin{aligned}&u=h(t)w(\zeta,\eta)\\ &\quad -h^{-2}(t)\{h(t)[h'(t)x+F_t+G_t]+4F_{\tau_+}G_{\tau_-}\},\\ &\zeta=h(t)x+F(\tau_+,t)+G(\tau_-,t),\\ &\eta=P(\tau_\varepsilon,t),\end{aligned} \quad (25)$$

where $P(\tau_\varepsilon,t)$, $h(t)$, $F(\tau_+,t)$ and $G(\tau_-,t)$ obey the constraints (23) and (24).

(B) $\sigma_z-\mathrm{i}\varepsilon\sigma_y=0$: In this case, (19) is satisfied identically. The general solution for $\sigma(y,z,t)$ reads

$$\sigma=R(\tau_\varepsilon,t), \quad (26)$$

where $R(\tau_\varepsilon,t)$ is an arbitrary complex differentiable function of τ_ε and t. With symbolic computation, solving (13) and (18) gives two sets of solutions about $P(\tau_\varepsilon,t)$ and $Q(\tau_\varepsilon,t)$ as

$$Q=S_1(\tau_\varepsilon),\quad P=S_1^2(\tau_\varepsilon)t+S_2(\tau_\varepsilon), \tag{27}$$

and

$$\begin{aligned}&Q=[K_1(\tau_\varepsilon)t+K_2(\tau_\varepsilon)]^{-1},\\&P=-[K_1^2(\tau_\varepsilon)t+K_1(\tau_\varepsilon)K_2(\tau_\varepsilon)]^{-1}+K_3(\tau_\varepsilon),\end{aligned} \tag{28}$$

where $S_i\ (i=1,2), K_j\ (j=1,2,3)$ are all arbitrary complex differentiable functions with respect to τ_ε, $S_1\neq 0$ and $K_1\neq 0$.

Therefore, two other transformations from (3) to (4) are obtained as follows:

$$\begin{aligned}&u=S_1(\tau_\varepsilon)w(\zeta,\eta)-S_1^{-1}(\tau_\varepsilon)R_t,\\&\zeta=S_1(\tau_\varepsilon)x+R(\tau_\varepsilon,t),\\&\eta=S_1^2(\tau_\varepsilon)t+S_2(\tau_\varepsilon),\end{aligned} \tag{29}$$

and

$$\begin{aligned}&u=[K_1(\tau_\varepsilon)t+K_2(\tau_\varepsilon)]^{-1}\\&\quad\cdot\left\{w(\zeta,\eta)+K_1(\tau_\varepsilon)x-[K_1(\tau_\varepsilon)t+K_2(\tau_\varepsilon)]^2R_t\right\},\\&\zeta=[K_1(\tau_\varepsilon)t+K_2(\tau_\varepsilon)]^{-1}x+R(\tau_\varepsilon,t),\\&\eta=-[K_1^2(\tau_\varepsilon)t+K_1(\tau_\varepsilon)K_2(\tau_\varepsilon)]^{-1}+K_3(\tau_\varepsilon).\end{aligned} \tag{30}$$

2.2. Relationship between ODBE and (1+1)-DBE

In this subsection, we construct the relationship between the ODBE and (1+1)-DBE. We integrate (4) with respect to ζ and obtain the forced Burgers equation

$$w_\eta+ww_\zeta-w_{\zeta\zeta}+f(\eta)=0, \tag{31}$$

where $f(\eta)$ is an arbitrary differentiable function of η. Then, by the transformation

$$\begin{aligned}&w=W(X,T)-\textstyle\int f(\eta)\mathrm{d}\eta,\\&X=\zeta+\textstyle\int(\int f(\eta)\mathrm{d}\eta)\,\mathrm{d}\eta,\\&T=\eta,\end{aligned} \tag{32}$$

(31) can be transformed into

$$W_T+WW_X-W_{XX}=0. \tag{33}$$

We thus have succeeded in transforming the (3+1)-DBE to a (1+1)-DBE.

3. Transformations from (2+1)-DBE to (1+1)-DBE

Following similar steps as in Section 2, we carry out symbolic computation once again and get two transformations from (2) to (4) which are exhibited as below:

$$\begin{aligned}&u=cw(\zeta,\eta)-c^{-2}[c\sigma_1'(t)y+c\sigma_0'(t)+\sigma_1^2(t)],\\&\zeta=cx+\sigma_1(t)y+\sigma_0(t),\\&\eta=c^2t,\end{aligned} \tag{34}$$

and

$$\begin{aligned}&u=(k_1t+k_2)^{-1}[w(\zeta,\eta)+k_1x]\\&\qquad-(k_1t+k_2)[\sigma_1'(t)y+(k_1t+k_2)\sigma_1^2(t)+\sigma_0'(t)],\\&\zeta=(k_1t+k_2)^{-1}x+\sigma_1(t)y+\sigma_0(t),\\&\eta=-(k_1^2t+k_1k_2)^{-1},\end{aligned} \tag{35}$$

where $c\neq 0$, $k_1\neq 0$ and k_2 are three arbitrary real constants, while $\sigma_0(t)$ and $\sigma_1(t)$ are two arbitrary real differentiable functions of t. Thus, transformation (34) or (35) plus transformation (32) eventually transforms the (2+1)-DBE to a (1+1)-DBE.

4. Remarks and Multi-Dimensional *N*-Shock-Wave-Like Solutions

In Sections 2 and 3, we have transformed the (2+1)- and (3+1)-DBEs to a (1+1)-DBE by ansatz (5), so that a subclass of the solutions of the two multi-dimensional Burgers equations is mapped onto (33). By virtue of the above transformations, it becomes possible to further investigate the (2+1)- and (3+1)-DBEs on the basis of the (1+1)-DBE which is completely integrable and has some special properties: the N-shock-wave solutions [22], similarity reductions to ordinary differential equations (ODEs) of Painlevé-type [34], Hopf-Cole transformation [34] and so on. Thus, we can naturally obtain new exact analytic solutions (especially the N-shock-wave-like solutions), similarity reductions to ODEs of Painlevé-type and Bäcklund transformations for the two multi-dimensional models. It is noted that symbolic computation which is a new branch of artificial intelligence has played a big role in our computing process. As a matter of fact, the (2+1)- and (3+1)-DBEs can also be transformed to other (1+1)-dimensional NLEEs, but that may not be helpful for constructing the N-shock-wave-like solutions. For many other multi-dimensional NLEEs ([36] and references therein), we can also transform them to lower-dimensional counterparts using ansatz (5) or the like, and then construct abundant exact analytic solutions, especially the *soliton-like solutions*.

Our current concern is to construct the multi-dimensional N-shock-wave-like solutions by virtue of the N-shock-wave solutions of (33) in the form [22]

$$W = \frac{2\sum_{i=1}^{N} a_i \mathrm{e}^{-a_i X + a_i^2 T}}{1+\sum_{i=1}^{N} \mathrm{e}^{-a_i X + a_i^2 T}}, \tag{36}$$

where N is an arbitrary positive integer, $a_i \neq 0$ are arbitrary real constants and $a_i \neq a_j$ $(i \neq j; i,j = 1,2,\cdots,N)$. [It is reasonable to assume the function $f(\eta)$ in transformation (32) to be zero, since transformation (32) has not substantial influence on the multi-dimensional N-shock-wave-like solutions to be constructed via expression (36).]

By the transformation (34), *a family of two-dimensional N-shock-wave-like solutions* to the (2+1)-DBE is given as

$$u_N^{(1)} = -\frac{c\sigma_1'(t)y + \sigma_1^2(t) + c\sigma_0'(t)}{c^2} + \frac{2c\sum_{i=1}^{N} a_i \mathrm{e}^{c^2 a_i^2 t - a_i[cx+\sigma_1(t)y+\sigma_0(t)]}}{1+\sum_{i=1}^{N} \mathrm{e}^{c^2 a_i^2 t - a_i[cx+\sigma_1(t)y+\sigma_0(t)]}}, \tag{37}$$

where c, $\sigma_0(t)$ and $\sigma_1(t)$ are defined as in Section 3. The one-shock-wave-like solutions given in [27, 28], which were expressed in a different way, correspond to some special cases of this family when $N = 1$ by selecting suitable $\sigma_0(t)$ and $\sigma_1(t)$. But to our knowledge, the cases for $N \geq 2$ in family (37) have not been reported.

By the transformations (25) and (29), *two families of three-dimensional N-shock-wave-like solutions* to the (3+1)-DBE are obtained as

$$u_N^{(2)} = -\frac{h(t)[h'(t)x + F_t + G_t] + 4F_{\tau_+}G_{\tau_-}}{h^2(t)} + \frac{2h(t)\sum_{i=1}^{N} a_i \mathrm{e}^{-a_i[h(t)x+F(\tau_+,t)+G(\tau_-,t)]+a_i^2 P(\tau_\varepsilon,t)}}{1+\sum_{i=1}^{N} \mathrm{e}^{-a_i[h(t)x+F(\tau_+,t)+G(\tau_-,t)]+a_i^2 P(\tau_\varepsilon,t)}}, \tag{38}$$

and

$$u_N^{(3)} = -\frac{R_t}{S_1(\tau_\varepsilon)} + 2S_1(\tau_\varepsilon) \cdot \frac{\sum_{i=1}^{N} a_i \mathrm{e}^{-a_i[S_1(\tau_\varepsilon)x+R(\tau_\varepsilon,t)]+a_i^2[S_1^2(\tau_\varepsilon)t+S_2(\tau_\varepsilon)]}}{1+\sum_{i=1}^{N} \mathrm{e}^{-a_i[S_1(\tau_\varepsilon)x+R(\tau_\varepsilon,t)]+a_i^2[S_1^2(\tau_\varepsilon)t+S_2(\tau_\varepsilon)]}}, \tag{39}$$

where ε, τ_ε, τ_+, τ_-, $h(t)$, $S_1(\tau_\varepsilon)$, $S_2(\tau_\varepsilon)$, $F(\tau_+,t)$, $G(\tau_-,t)$, $P(\tau_\varepsilon,t)$ and $R(\tau_\varepsilon,t)$ are defined as in Section 2. In family (38), the functions $h(t)$, $F(\tau_+,t)$, $G(\tau_-,t)$ and $P(\tau_\varepsilon,t)$ satisfy the constraints (23) and (24). The one-shock-wave-like solutions presented in [29] are only two special cases for $N = 1$ in family (38) when $h(t)$, $F(\tau_+,t)$, $G(\tau_-,t)$ and $P(\tau_\varepsilon,t)$ are selected suitably. As far as we know, family (39) and other cases in family (38) are exhibited here for the first time.

Two other families of multi-dimensional N-shock-wave-like solutions by the transformations (30) and (35) will blow up in a finite time, so we do not present these two families, which are both singular. In addition, we find that family (39) is complex, which is only of mathematical interest because the Burgers field is usually assumed to be real in physics [37], while family (38) is not always complex, which depends on the functions $h(t)$, $F(\tau_+,t)$, $G(\tau_-,t)$ and $P(\tau_\varepsilon,t)$. By choosing

$$h(t) = \delta,\ F(\tau_+,t) = \frac{\tau_+}{2}[\lambda(t) + \mathrm{i}\mu(t)] + \rho(t), \tag{40}$$

$$P(\tau_\varepsilon,t) = \delta^2 t, \quad G(\tau_-,t) = \frac{\tau_-}{2}[\lambda(t) - \mathrm{i}\mu(t)], \tag{41}$$

where $\delta \neq 0$ is an arbitrary real constant, $\lambda(t)$, $\mu(t)$ and $\rho(t)$ are three arbitrary real functions of t, we exhibit a real subset of family (38), which may be physically interesting, written as

$$u_N^{(4)} = -\frac{\delta[\mu'(t)y + \lambda'(t)z + \rho'(t)] + \lambda^2(t) + \mu^2(t)}{\delta^2} + \frac{2\delta\sum_{i=1}^{N} a_i \mathrm{e}^{\delta^2 a_i^2 t - a_i[\delta x + \lambda(t)z + \mu(t)y + \rho(t)]}}{1+\sum_{i=1}^{N} \mathrm{e}^{\delta^2 a_i^2 t - a_i[\delta x + \lambda(t)z + \mu(t)y + \rho(t)]}}. \tag{42}$$

5. Discussion and Summary

Differing from the N-shock-wave solutions obtained by the bilinear transformation method in [8, 9, 11], families (37) and (42) are both real and contain several arbitrary functions which result from the reduction of space dimensionality. For these two families, we would like to concentrate ourselves on the non-traveling-wave effects caused by the arbitrary functions and inelastic interaction of shock waves.

The detailed application of families (37) and (42) requires a judicious choice of the free parameters and functions involved in the solutions [37]. Hereby for the picture drawing and qualitative analysis, we choose

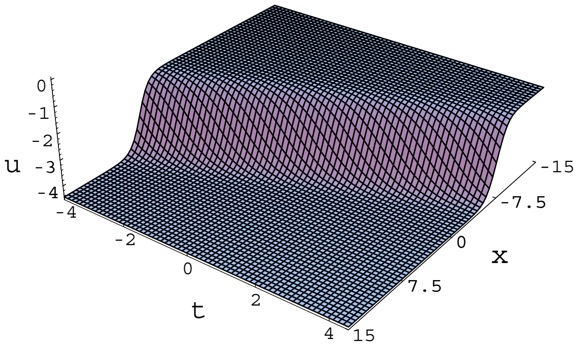

Fig. 1. The well-known Taylor shock profile via expression (43) at $y=-1.5$, to be compared with Figs. 2 and 3. The related parameters and functions are chosen as $c=0.8$, $a_1=-1.8$, $\sigma_0(t)=2$ and $\sigma_1(t)=1$.

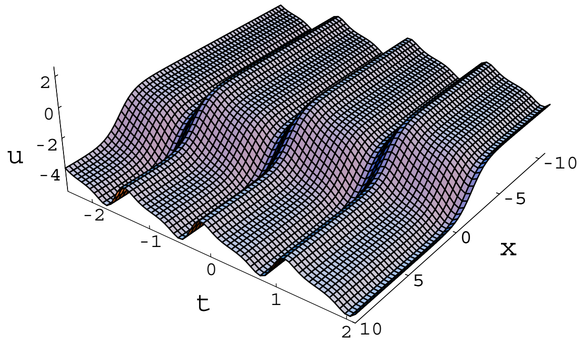

Fig. 2. The first one-shock-wave-like solution surface for expression (43) at $y=-1.5$ with the same parameters and functions as in Fig. 1 except that $\sigma_0(t)=0.06\mathrm{e}^{0.5-\sin(5t)}$.

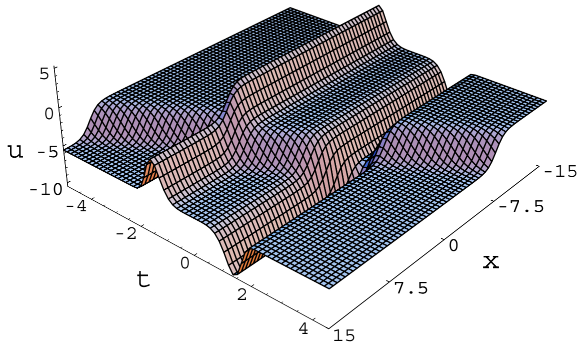

Fig. 3. The second one-shock-wave-like solution surface for expression (43) at $y=-1.5$ with the same parameters and functions as in Fig. 1 except that $a_1=-2.5$ and $\sigma_0(t)=-2.5\mathrm{e}^{-0.1t^6}$.

some values for those parameters and functions in line with the nonuniform background, non-traveling-wave features in the two multi-dimensional Burgers equations. In reality, there is no essential difference between families (37) and (42) except the space dimensionality. Therefore, in the following discussion, we will focus on family (37), all the features of which can also be applicable to family (42).

For $N=1$, family (37) corresponds to the two-dimensional one-shock-wave-like solution

$$u_1^{(1)}=ca_1\tanh\left\{\frac{c^2a_1^2t-a_1[cx+\sigma_1(t)y+\sigma_0(t)]}{2}\right\}-\frac{c\sigma_1'(t)y+\sigma_1^2(t)+c\sigma_0'(t)-c^3a_1}{c^2}, \tag{43}$$

from which it can be seen that the phase and equilibrium position of the shock wave are both influenced by two arbitrary functions $\sigma_0(t)$ and $\sigma_1(t)$.

Figure 1, with the constant values of parameters and functions in its caption, presents a traveling shock wave transverse along the y axis via expression (43), which is the well-known Taylor shock profile. In comparison, Figs. 2 and 3 provide us with two samples of one-shock-wave-like solution surfaces for expression (43), both *beyond the traveling waves*. In Fig. 2, the shock wave progresses along the propagation direction with the fluctuations of phase and equilibrium position. In Fig. 3, the phase and equilibrium position of shock wave undergo a transitory change simultaneously, after that the initial profile recurs. It is concluded that Figs. 2 and 3 exhibit two non-traveling-wave features: (1) *ceaselessly varying velocities of propagation* and (2) *phase shifts in a localized temporal area*. In principle, more non-traveling-wave features hide in expression (43) for the arbitrariness of $\sigma_0(t)$ and $\sigma_1(t)$. Thus, expression (43) could more realistically describe some situations like the motion of upper-surface perturbations in a shallow viscous fluid [5,38], nonplanar structures of multi-dimensional shocks modified by diffusing cosmic rays [6,10,39] and small electromagnetic perturbations in a saturated ferrite in the presence of an external magnetic field [8,11], as shown in Figs. 2 and 3, the *non-traveling-wave effects* of which may be observed in the relevant fields.

For $N\geq 2$, family (37) could describe the coalescence phenomena of N traveling/non-traveling shock waves. Figure 4 shows that two traveling waves which are both Taylor-type shock profiles along opposite propagation directions coalesce into one large shock wave, the amplitude of which amounts to two initial amplitudes. In contrast, Fig. 5 displays the coalescence process of two Taylor-type shock waves along

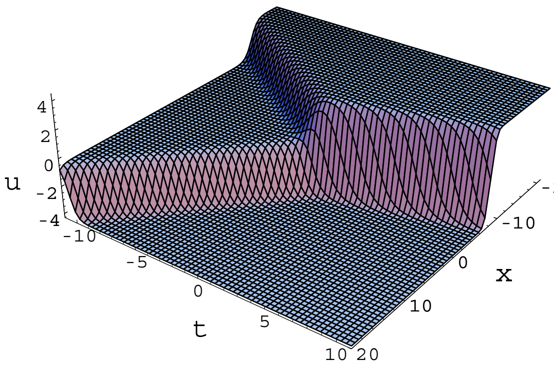

Fig. 4. The first two-shock-wave solution surface via expression (37) at $y = 1$, to be compared with Figs. 5 and 6. The related parameters and functions are chosen as $N = 2$, $c = 1$, $a_1 = 1.5$, $a_2 = -1.8$, $\sigma_0(t) = 0.25t + 1$ and $\sigma_1(t) = 0.5$.

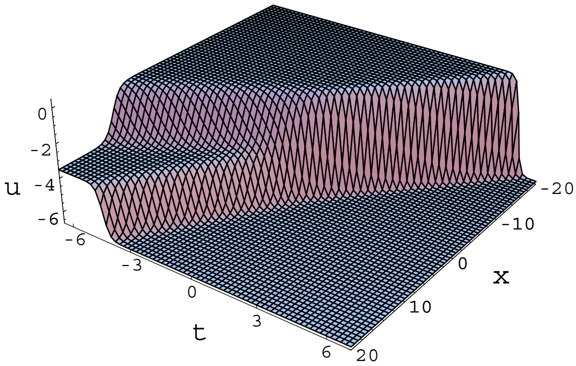

Fig. 5. The second two-shock-wave solution surface via expression (37) at $y = -1$ with the same parameters and functions as in Fig. 4 except that $a_1 = -1.5$ and $a_2 = -3$. This picture is for comparison with Figs. 4 and 7.

the same propagation direction but with different velocities. When choosing nonconstant values for $\sigma_0(t)$ and $\sigma_1(t)$, Figs. 6 and 7 present two samples of two-shock-wave-like solution surfaces for family (37), also *beyond the traveling waves*. It is easy to find that Figs. 6 and 7, respectively, modify Figs. 4 and 5 with *ceaselessly varying velocities of propagation and rolling equilibrium positions* for shock waves. With the increase of N in family (37), the coalescence behaviors of shock waves will become more complicated, as illustrated in Fig. 8 which reflects the interaction of three non-traveling shock waves.

It is well-known that there is no exchange of physical quantities (no change of shapes and velocities) among solitons except for the phase shifts in the elastically colliding process [40]. From family (37) and the graphical analysis in Figs. 4 – 8, we can draw the con-

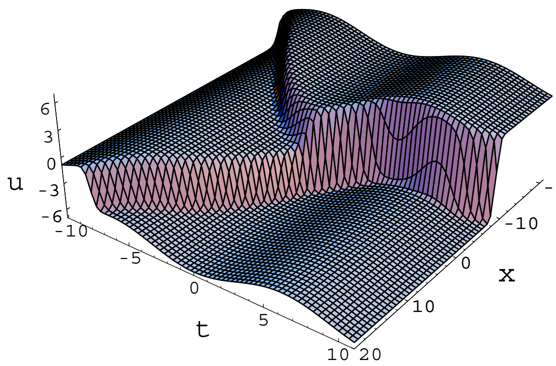

Fig. 6. The first two-shock-wave-like solution surface for expression (37) at $y = 1$ with the same parameters and functions as in Fig. 4 except that $c = 1.4$ and $\sigma_0(t) = 3\sin(0.5t) + 0.5t + 1$.

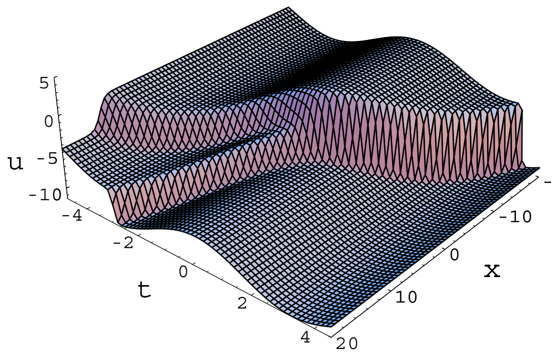

Fig. 7. The second two-shock-wave-like solution surface for expression (37) at $y = -1$ with the same parameters and functions as in Fig. 5 except that $c = 1.5$ and $\sigma_0(t) = 3\sin(t + 3) + 0.5t + 1$.

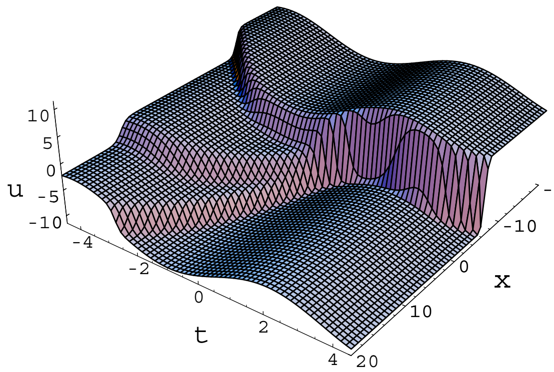

Fig. 8. A three-shock-wave-like solution surface via expression (37) at $y = 0.5$ with the related parameters and functions chosen as $N = 3$, $c = 1$, $a_1 = -2$, $a_2 = 2.8$, $a_3 = -4$, $\sigma_0(t) = 2.5\sin(t + 1) + 0.5t$ and $\sigma_1(t) = 0.2$.

clusion that the interaction of traveling/non-traveling shock waves for the (2+1)-DBE is inelastic. Another

t	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8
−15	2.88000	2.88000	4.00000	6.60000	6.00000	9.24000	9.00000	13.6000
−10	2.88000	2.88000	4.00000	6.60000	6.00000	9.24000	9.00000	13.6000
−5	2.88000	2.88000	4.00000	6.60000	6.00000	9.24000	9.00000	13.6000
0	2.88000	2.88000	4.00000	6.60000	6.00000	9.24000	9.00000	13.6000
5	2.88000	2.88000	4.00000	6.60000	6.00000	9.24000	9.00000	13.6000
10	2.88000	2.88000	4.00000	6.60000	6.00000	9.24000	9.00000	13.6000
15	2.88000	2.88000	4.00000	6.60000	6.00000	9.24000	9.00000	13.6000

Table 1. Calculated values of the conserved quantity $I = \int_{-\infty}^{+\infty} |q| \mathrm{d}x$, corresponding to the choices of parameters and functions in Figures 1 – 8.

noticeable observation in the interaction process is that the total amplitudes of initial shock waves are equal to the single one after interaction. By numerical calculations, it can be testified that the total transverse mass of the (2+1)-DBE along the y axis is conserved before and after interaction (see Table 1), that is,

$$I = \int_{-\infty}^{+\infty} |q(x,\bar{y},t)| \mathrm{d}x = \text{ constant}, \qquad (44)$$

where $q = u_x$ and $\bar{y}$ is a fixed value. Similarly, the total transverse mass of the system along the x axis also keeps unchanged.

Since the inelastic interaction of solitons is of current interest in both theory and experiment [12 – 21], special attention should be paid to the cases for $N \geq 2$ in family (37) and those *inelastically interacting properties and non-traveling-wave effects* in Figs. 4 – 8, which are expected to be observed in viscous fluids, astrophysics, electromagnetics and other fields. On the other hand, family (37), when $N \geq 2$, might be very useful for explaining the mergence of wavefronts produced by the spatial and temporal modulations of original uniformly traveling waves on a flowing film [23], interaction of impurity solitons in hydrodynamics [24], coalescence of electromagnetic waves in a saturated ferromagnetic medium [8, 11], inelastic collisions of particles in a sticky gas [25] and nontrivial coalescence behaviors of passive random walkers on a growing surface [26].

All the pictures above are drawn at a fixed "y", in order to clearly illustrate the propagation process of shock waves with the development of time. If those pictures are taken at a fixed "x", as a matter of fact, similar effects will be observed.

Finally, we summarize this paper as follows:

1. Under investigation in this paper are two important multi-dimensional nonlinear models, i. e., the (2+1)- and (3+1)-DBEs, which arise in many situations such as the motion of upper-surface perturbations of a shallow viscous fluid, nonplanar cosmic-ray-modified shock structures and propagation of electromagnetic waves in a saturated ferrite.

2. With the aid of symbolic computation, the transformations from the (2+1)- and (3+1)-DBEs to a (1+1)-DBE have been constructed by ansatz (5). Via the obtained transformations, we have specially presented several families of multi-dimensional N-shock-wave-like solutions, of which some previously published solutions turn out to be special cases. For many other multi-dimensional NLEEs, it is also feasible to transform them to lower-dimensional counterparts by use of ansatz (5) or the like.

3. In view of current interest in the inelastic interaction of solitons, we have discussed the inelastically interacting properties of multi-dimensional shock waves based on above figures for several sample solutions. Additionally, we have pointed out some non-traveling-wave effects, the underlying mechanisms of which have a strong bearing on the higher space dimensionality of the (2+1)- and (3+1)-DBEs. It is hoped that those solutions and effects are useful for future studies in some fields.

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